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Do Minkowski averages get progressively more convex?

*Les moyennes de Minkowski deviennent-elles progressivement plus convexes ?*Matthieu Fradelizi^{a,1}, Mokshay Madiman^{b,2}, Arnaud Marsiglietti^{c,3},
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ABSTRACT

Let us define, for a compact set $A \subset \mathbf{R}^n$, the Minkowski averages of A :

$$A(k) = \left\{ \frac{a_1 + \dots + a_k}{k} : a_1, \dots, a_k \in A \right\} = \frac{1}{k} \underbrace{(A + \dots + A)}_{k \text{ times}}.$$

We study the monotonicity of the convergence of $A(k)$ towards the convex hull of A , when considering the Hausdorff distance, the volume deficit and a non-convexity index of Schneider as measures of convergence. For the volume deficit, we show that monotonicity fails in general, thus disproving a conjecture of Bobkov, Madiman and Wang. For Schneider's non-convexity index, we prove that a strong form of monotonicity holds, and for the Hausdorff distance, we establish that the sequence is eventually nonincreasing.

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R É S U M É

Pour tout ensemble compact $A \subset \mathbf{R}^n$, définissons ses moyennes de Minkowski par

$$A(k) = \left\{ \frac{a_1 + \dots + a_k}{k} : a_1, \dots, a_k \in A \right\} = \frac{1}{k} \underbrace{(A + \dots + A)}_{k \text{ fois}}.$$

Nous étudions la monotonie de la convergence de $A(k)$ vers l'enveloppe convexe de A , mesurée par la distance de Hausdorff, le déficit volumique et par l'indice de non-convexité

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de Schneider. Pour le déficit volumique, nous démontrons que la propriété de monotonie n'est pas satisfaite en général, réfutant ainsi une conjecture de Bobkov, Madiman et Wang. Pour l'index de non-convexité de Schneider, nous montrons une propriété renforcée de monotonie, tandis que, pour la distance de Hausdorff, nous établissons que la suite est décroissante à partir d'un certain rang.

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Version française abrégée

L'objectif de cette note est d'annoncer et de démontrer une partie des résultats obtenus dans [3] qui portent sur l'étude de la monotonie de la suite $(A(k))_{k \geq 1}$ définie en (1), mesurée à travers différentes mesures de non-convexité. Intuitivement, les ensembles $A(k)$ deviennent de plus en plus convexes au fur et à mesure que k croît. Cette intuition est précisée dans [7,2] où il est démontré que la suite $(A(k))$ converge vers son enveloppe convexe en distance de Hausdorff d_H .

L'origine de notre étude provient d'une conjecture de Bobkov, Madiman et Wang [1], qui affirme que la suite $(\Delta(A(k)))_{k \geq 1}$ est décroissante, où

$$\Delta(A) := \text{Vol}_n(\text{conv}(A) \setminus A) = \text{Vol}_n(\text{conv}(A)) - \text{Vol}_n(A)$$

désigne le déficit volumique d'un ensemble compact de \mathbf{R}^n . Ici, Vol_n représente la mesure de Lebesgue dans \mathbf{R}^n et $\text{conv}(A)$ désigne l'enveloppe convexe de A . Nous réfutons cette conjecture en exhibant un contre-exemple explicite en dimension supérieure ou égale à 12. Le contre-exemple est la réunion de deux ensembles convexes inclus dans des sous-espaces de dimension (presque) moitié de l'espace ambiant (voir Fig. 1). Nous démontrons aussi la validité de la conjecture en dimension 1 en adaptant une démonstration de [4] sur le cardinal de sommes d'entiers; cela a aussi été observé indépendamment par F. Barthe. La conjecture reste ouverte en dimension n , pour $1 < n < 12$.

De manière analogue à la conjecture de Bobkov–Madiman–Wang, nous étudions la monotonie de la suite $(c(A(k)))_{k \geq 1}$, où c est l'index de non-convexité de Schneider [6] défini par

$$c(A) := \inf\{\lambda \geq 0 : A + \lambda \text{ conv}(A) \text{ est convexe}\}.$$

Contrairement au déficit volumique, la suite $(c(A(k)))$ est strictement décroissante, à moins que $A(k)$ soit déjà convexe. Plus précisément nous montrons que pour tout ensemble compact A de \mathbf{R}^n et tout $k \in \mathbb{N}^*$

$$c(A(k+1)) \leq \frac{k}{k+1} c(A(k)).$$

En outre, nous étudions dans [3] la monotonie de $A(k)$, mesurée par d'autres mesures de non-convexité. Ainsi, nous montrons que si l'on pose

$$d(A) = d_H(A, \text{conv}(A)) = \inf\{r > 0 : \text{conv}(A) \subset A + rB_2^n\},$$

où B_2^n est la boule euclidienne centrée en 0 de rayon 1, alors pour tout compact A de \mathbf{R}^n et pour $k \geq c(A)$,

$$d(A(k+1)) \leq \frac{k}{k+1} d(A(k)).$$

1. Introduction

This note announces and proves some of the results obtained in [3]. Let us denote for a compact set $A \subset \mathbf{R}^n$ and for a positive integer k ,

$$A(k) = \left\{ \frac{a_1 + \dots + a_k}{k} : a_1, \dots, a_k \in A \right\} = \frac{1}{k} \underbrace{(A + \dots + A)}_{k \text{ times}}. \quad (1)$$

Denoting by $\text{conv}(A)$ the convex hull of A , and by

$$d(A) := \inf\{r > 0 : \text{conv}(A) \subset A + rB_2^n\}$$

the Hausdorff distance between a set A and its convex hull, it is a classical fact (proved independently by [7,2] in 1969, and often called the Shapley–Folkman–Starr theorem) that $A(k)$ converges in Hausdorff distance to $\text{conv}(A)$ as $k \rightarrow \infty$. Furthermore [7,2] also determined the rate of convergence: it turns out that $d(A(k)) = O(1/k)$ for any compact set A . For sets of nonempty interior, this convergence of Minkowski averages to the convex hull can also be expressed in terms of the volume deficit $\Delta(A)$ of a compact set A in \mathbf{R}^n , which is defined as:

$$\Delta(A) := \text{Vol}_n(\text{conv}(A) \setminus A) = \text{Vol}_n(\text{conv}(A)) - \text{Vol}_n(A),$$

where Vol_n denotes Lebesgue measure in \mathbf{R}^n . It was shown by [2] that if A is compact with nonempty interior, then the volume deficit of $A(k)$ also converges to 0; more precisely, $\Delta(A(k)) = O(1/k)$ for any compact set A with nonempty interior.

Our original motivation came from a conjecture made by Bobkov, Madiman and Wang [1]:

Conjecture 1. (See [1].) Let A be a compact set in \mathbf{R}^n for some $n \in \mathbb{N}$, and let $A(k)$ be defined as in (1). Then the sequence $\Delta(A(k))$ is non-increasing in k , or equivalently, $\{\text{Vol}_n(A(k))\}_{k \geq 1}$ is non-decreasing.

We show that Conjecture 1 fails to hold in general, even for moderately high dimension.

Theorem 2. Conjecture 1 is false in \mathbf{R}^n for $n \geq 12$, and true for \mathbf{R}^1 .

Notice that Conjecture 1 remains open for $1 < n < 12$. In particular, the arguments presented in this note do not seem to work. In analogy with Conjecture 1, we also consider whether one can have monotonicity of $\{c(A(k))\}_{k \geq 1}$, where c is a non-convexity index defined by Schneider [6] as follows:

$$c(A) := \inf\{\lambda \geq 0 : A + \lambda \text{conv}(A) \text{ is convex}\}.$$

A nice property of Schneider's index is that it is affine-invariant, i.e., $c(TA + x) = c(A)$ for any nonsingular linear map T on \mathbf{R}^n and any $x \in \mathbf{R}^n$.

Contrary to the volume deficit, we prove that Schneider's non-convexity index c satisfies a strong kind of monotonicity in any dimension.

Theorem 3. Let A be a compact set in \mathbf{R}^n and $k \in \mathbb{N}^*$. Then

$$c(A(k+1)) \leq \frac{k}{k+1} c(A(k)).$$

Finally, we also prove that eventually, for $k \geq c(A)$, the Hausdorff distance between $A(k)$ and $\text{conv}(A)$ is also strongly decreasing.

Theorem 4. Let A be a compact set in \mathbf{R}^n and $k \geq c(A)$ be an integer. Then

$$d(A(k+1)) \leq \frac{k}{k+1} d(A(k)).$$

Moreover, Schneider proved in [6] that $c(A) \leq n$ for every compact subset A of \mathbf{R}^n . It follows that the eventual monotonicity of the sequence $d(A(k))$ holds true for $k \geq n$.

It is natural to ask what the relationship is in general between convergence of c , Δ and d to 0, for arbitrary sequences (C_k) of compact sets. In fact, none of these three notions of approach to convexity are comparable with each other in general. To see why, observe that while c is scaling-invariant, neither Δ nor d are; so it is easy to construct examples of sequences (C_k) such that $c(C_k) \rightarrow 0$ but $\Delta(C_k)$ and $d(C_k)$ remain bounded away from 0. The same argument enables us to construct examples of sequences (C_k) such that $c(C_k)$ remain bounded away from 0, whereas $\Delta(C_k)$ and $d(C_k)$ converge to 0. Furthermore, $\Delta(C_k)$ remains bounded away from 0 for any sequence C_k of finite sets, whereas $c(C_k)$ and $d(C_k)$ could converge to 0 if the finite sets form a finer and finer grid filling out a convex set. An example where $\Delta(C_k) \rightarrow 0$ but both $c(C_k)$ and $d(C_k)$ are bounded away from 0 is given by taking a 3-point set with 2 of the points getting arbitrarily closer but staying away from the third. One can obtain further relationships between these measures of non-convexity if further conditions are imposed on the sequence C_k ; details may be found in [3].

The rest of this note is devoted to the examination of whether $A(k)$ becomes progressively more convex as k increases, when measured through the functionals Δ , d and c . The concluding section contains some additional discussion.

2. The behavior of volume deficit

We prove Theorem 2 in this section. We start by constructing a counterexample to the conjecture in \mathbf{R}^n , for $n \geq 12$. Let F be a p -dimensional subspace of \mathbf{R}^n , where $p \in \{1, \dots, n-1\}$. Let us consider $A = I_1 \cup I_2$, where $I_1 \subset F$ and $I_2 \subset F^\perp$, where F^\perp denotes the orthogonal complement of F . One has (see Fig. 1):

$$A + A = 2I_1 \cup (I_1 \times I_2) \cup 2I_2,$$

$$A + A + A = 3I_1 \cup (2I_1 \times I_2) \cup (I_1 \times 2I_2) \cup 3I_2.$$

Notice that

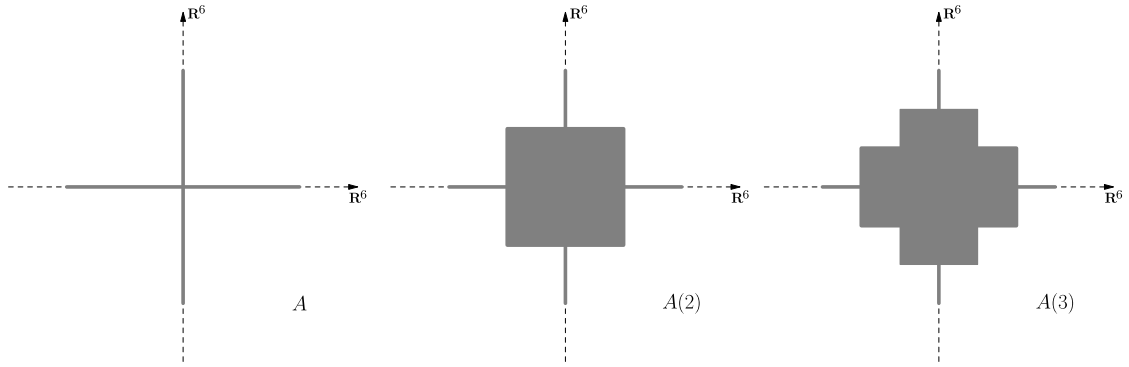


Fig. 1. A counterexample in \mathbb{R}^{12} .

$$\text{Vol}_n(A + A) = \text{Vol}_p(I_1)\text{Vol}_{n-p}(I_2),$$

$$\text{Vol}_n(A + A + A) = \text{Vol}_p(I_1)\text{Vol}_{n-p}(I_2)(2^p + 2^{n-p} - 1).$$

Thus, $\text{Vol}_n(A(3)) \geq \text{Vol}_n(A(2))$ if and only if

$$2^p + 2^{n-p} - 1 \geq \left(\frac{3}{2}\right)^n. \quad (2)$$

Notice that inequality (2) does not hold when $n \geq 12$ and $p = \lceil \frac{n}{2} \rceil$.

For \mathbb{R}^1 , the conjecture may be proved by adapting a proof of [4] on cardinality of integer sumsets; this was also independently observed by F. Barthe. Let $k \geq 1$. Set $S = A_1 + \dots + A_k$ and for $i \in [k]$, let $a_i = \min A_i$, $b_i = \max A_i$,

$$S_i = \sum_{j \in [k] \setminus \{i\}} A_j,$$

$s_i = \sum_{j < i} a_j + \sum_{j > i} b_j$, $S_i^- = \{x \in S_i; x \leq s_i\}$ and $S_i^+ = \{x \in S_i; x > s_i\}$. For all $i \in [k-1]$, one has

$$S \supset (a_i + S_i^-) \cup (b_{i+1} + S_{i+1}^+).$$

Since $a_i + s_i = \sum_{j \leq i} a_j + \sum_{j > i} b_j = b_{i+1} + s_{i+1}$, the above union is a disjoint union. Thus for $i \in [k-1]$

$$\text{Vol}_1(S) \geq \text{Vol}_1(a_i + S_i^-) + \text{Vol}_1(b_{i+1} + S_{i+1}^+) = \text{Vol}_1(S_i^-) + \text{Vol}_1(S_{i+1}^+).$$

Notice that $S_1^- = S_1$ and $S_k^+ = S_k \setminus \{s_k\}$, thus adding the above $k-1$ inequalities, we obtain

$$\begin{aligned} (k-1)\text{Vol}_1(S) &\geq \sum_{i=1}^{k-1} (\text{Vol}_1(S_i^-) + \text{Vol}_1(S_{i+1}^+)) = \text{Vol}_1(S_1^-) + \text{Vol}_1(S_k^+) + \sum_{i=2}^{k-1} \text{Vol}_1(S_i) \\ &= \sum_{i=1}^k \text{Vol}_1(S_i). \end{aligned}$$

Now taking all the sets $A_i = A$, and dividing through by $k(k-1)$, we see that we have established [Conjecture 1](#) in dimension 1.

3. The behavior of Schneider's non-convexity index and the Hausdorff distance

We establish [Theorems 3 and 4](#) in this section. This relies crucially on the elementary observations that $\text{conv}(A+B) = \text{conv}(A) + \text{conv}(B)$ and $(t+s)\text{conv}(A) = t\text{conv}(A) + s\text{conv}(A)$ for any $t, s > 0$ and any compact sets A, B .

Proof of Theorem 3. Denote $\lambda = c(A(k))$. Since $\text{conv}(A(k)) = \text{conv}(A)$, from the definition of c , one knows that $A(k) + \lambda \text{conv}(A) = \text{conv}(A) + \lambda \text{conv}(A) = (1+\lambda)\text{conv}(A)$. Using that $A(k+1) = \frac{A}{k+1} + \frac{k}{k+1}A(k)$, one has

$$\begin{aligned} A(k+1) + \frac{k}{k+1}\lambda \text{conv}(A) &= \frac{A}{k+1} + \frac{k}{k+1}A(k) + \frac{k}{k+1}\lambda \text{conv}(A) \\ &= \frac{A}{k+1} + \frac{k}{k+1}\text{conv}(A) + \frac{k}{k+1}\lambda \text{conv}(A) \end{aligned}$$

$$\begin{aligned}
&\supset \frac{\text{conv}(A)}{k+1} + \frac{k}{k+1}A(k) + \frac{k}{k+1}\lambda \text{conv}(A) \\
&= \frac{\text{conv}(A)}{k+1} + \frac{k}{k+1}(1+\lambda) \text{conv}(A) \\
&= \left(1 + \frac{k}{k+1}\lambda\right) \text{conv}(A).
\end{aligned}$$

Since the other inclusion is trivial, we deduce that $A(k+1) + \frac{k}{k+1}\lambda \text{conv}(A)$ is convex, which proves that

$$c(A(k+1)) \leq \frac{k}{k+1}\lambda = \frac{k}{k+1}c(A(k)). \quad \square$$

Proof of Theorem 4. Let $k \geq c(A)$, then, from the definitions of $c(A)$ and $d(A(k))$, one has

$$\begin{aligned}
\text{conv}(A) &= \frac{A}{k+1} + \frac{k}{k+1} \text{conv}(A) \subset \frac{A}{k+1} + \frac{k}{k+1} (A(k) + d(A(k))B_2^n) \\
&= A(k+1) + \frac{k}{k+1} d(A(k))B_2^n.
\end{aligned}$$

We conclude that

$$d(A(k+1)) \leq \frac{k}{k+1} d(A(k)). \quad \square$$

4. Discussion

- (i) By repeated application of Theorem 3, it is clear that the convergence of $c(A(k))$ is at a rate $O(1/k)$ for any compact set $A \subset \mathbb{R}^n$; this observation appears to be new. In [3], we study the question of the monotonicity of $A(k)$, as well as convergence rates, when considering several different ways to measure non-convexity, including some not mentioned in this note.
- (ii) Some of the results in this note are of interest when one is considering Minkowski sums of different compact sets, not just sums of A with copies of itself. Indeed, the original conjecture of [1] was of this form, and would have provided a strengthening of the classical Brunn–Minkowski inequality for more than 2 sets; of course, that conjecture is false since the weaker Conjecture 1 is false. Nonetheless we do have some related observations in [3]; for instance, it turns out that in general dimension, for compact sets A_1, \dots, A_k ,

$$\text{Vol}_n \left(\sum_{i=1}^k A_i \right) \geq \frac{1}{k-1} \sum_{i=1}^k \text{Vol}_n \left(\sum_{j \in [k] \setminus \{i\}} A_j \right).$$

For convex sets B_i , an even stronger fact is true (that this is stronger may not be immediately obvious), but it follows from well-known results, see, e.g., [5]:

$$\text{Vol}_n(B_1 + B_2 + B_3) + \text{Vol}_n(B_1) \geq \text{Vol}_n(B_1 + B_2) + \text{Vol}_n(B_1 + B_3).$$

- (iii) There is a variant of the strong monotonicity of Schneider's index when dealing with different sets. If A, B, C are subsets of \mathbb{R}^n , then it is shown in [3] (by a similar argument to that used for Theorem 3) that $c(A+B+C) \leq \max\{c(A+B), c(B+C)\}$.

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